

AN ITÔ FORMULA IN THE SPACE OF TEMPERED DISTRIBUTIONS

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ABSTRACT. We extend the Itô formula [14, Theorem 2.3] for semimartingales with rcll paths. We also comment on Local time process of such semimartingales. We apply the Itô formula to Lévy processes to obtain existence of solutions to certain classes of stochastic differential equations in the Hermite-Sobolev spaces.

1. INTRODUCTION

Itô formula is an important result in stochastic calculus and has been studied in quite generality, starting from real valued processes to processes taking values in Nuclear spaces ([2, 4, 7–10, 13–15, 19]).

Let $\mathcal{S}(\mathbb{R}^d)$ denote the space of real valued rapidly decreasing smooth functions on \mathbb{R}^d and let $\mathcal{S}'(\mathbb{R}^d)$ denote the dual space, i.e. the space of tempered distributions. For $p \in \mathbb{R}$, let $\mathcal{S}_p(\mathbb{R}^d)$ denote the Hermite-Sobolev spaces and for $x \in \mathbb{R}^d$, let τ_x denote the translation operators (see definitions in Section 2). Given $\phi \in \mathcal{S}_{-p}(\mathbb{R}^d)$ and an \mathbb{R}^d valued continuous semimartingale $X_t = (X_t^1, \dots, X_t^d)$, we have the following Itô formula (see [14, Theorem 2.3])

Theorem 1.1. *$\{\tau_{X_t}\phi\}$ is an $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued continuous semimartingale and we have the equality in $\mathcal{S}_{-p-1}(\mathbb{R}^d)$, a.s.*

$$\tau_{X_t}\phi = \tau_{X_0}\phi - \sum_{i=1}^d \int_0^t \partial_i \tau_{X_s}\phi dX_s^i + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \partial_{ij}^2 \tau_{X_s}\phi d[X^i, X^j]_s, t \geq 0.$$

This result has been used in [15] to show existence of solution of some stochastic differential equations in $\mathcal{S}'(\mathbb{R}^d)$. The aim of the current paper is to prove the result for semimartingales $\{X_t\}$ with rcll (right continuous with left limits) paths.

A version of this Itô formula was also proved in [19, Theorem III.1] with equality in \mathcal{S}' . In [8, Theorem 3], the author has proved this formula for twice continuously (Fréchet) differentiable function while dealing with a single Hilbert space. Note that derivatives of tempered distributions may not be in the same Hermite Sobolev space as the original one. Using regularization, the result [8, Theorem 3] was also proved in [11, Theorem 8] in the case of an E' valued continuous martingale, where E is a countably Hilbertian Nuclear space.

In Section 2, we recall the countably Hilbertian topology defined on $\mathcal{S}(\mathbb{R}^d)$ which gives rise to the Hermite-Sobolev spaces $\mathcal{S}_p(\mathbb{R}^d)$.

2010 *Mathematics Subject Classification.* Primary: 60H05; Secondary: 60H10, 60H15.

Key words and phrases. Hermite-Sobolev spaces, Tempered distributions, \mathcal{S}' valued processes, Itô formula, Local times, Stochastic Integral, Lévy processes.

In Section 3, we provide the construction of the stochastic integral of an $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued norm-bounded predictable process $\{G_t\}$ with respect to a real valued semimartingale $\{X_t\}$ from the first principles. Since the semimartingale is real valued, this procedure is simpler than the Hilbert valued stochastic integration described in [9, Chapter 4 and 5]. We note that for any $\phi \in \mathcal{S}(\mathbb{R}^d)$, a.s.

$$\left\langle \int_0^t G_s dX_s, \phi \right\rangle = \int_0^t \langle G_s, \phi \rangle dX_s, t \geq 0.$$

We exploit this property to prove an Itô formula (see Theorem 4.5). As an application, we consider an one-dimensional Lévy process X and show the existence of a solution of a stochastic differential equation (Theorem 4.7) in the Hermite-Sobolev spaces. This is similar to the solution obtained in [15] for continuous processes X .

2. TOPOLOGIES ON \mathcal{S} AND \mathcal{S}'

Let $\mathcal{S}(\mathbb{R}^d)$ be the space of smooth rapidly decreasing \mathbb{R} -valued functions on \mathbb{R}^d with the topology given by L. Schwartz (see [18]) and let $\mathcal{S}'(\mathbb{R}^d)$ be the dual space, known as the space of tempered distributions. Let $\mathcal{S}_p(\mathbb{R}^d)$ be the completion of $(\mathcal{S}(\mathbb{R}^d), \|\cdot\|_p)$ for any $p \in \mathbb{R}$ (see [4, Chapter 1.3] for the notations). The spaces $\mathcal{S}_p(\mathbb{R}^d), p \in \mathbb{R}$ are separable Hilbert spaces and are known as the Hermite-Sobolev spaces. We write $\mathcal{S}, \mathcal{S}', \mathcal{S}_p$ instead of $\mathcal{S}(\mathbb{R}), \mathcal{S}'(\mathbb{R}), \mathcal{S}_p(\mathbb{R})$.

Note that $\mathcal{S}_0(\mathbb{R}^d) = \mathcal{L}^2(\mathbb{R}^d)$ and for $p > 0$, $(\mathcal{S}_{-p}(\mathbb{R}^d), \|\cdot\|_{-p})$ is dual to $(\mathcal{S}_p(\mathbb{R}^d), \|\cdot\|_p)$. Furthermore,

$$\mathcal{S}(\mathbb{R}^d) = \bigcap_{p \in \mathbb{R}} (\mathcal{S}_p(\mathbb{R}^d), \|\cdot\|_p), \quad \mathcal{S}'(\mathbb{R}^d) = \bigcup_{p \in \mathbb{R}} (\mathcal{S}_p(\mathbb{R}^d), \|\cdot\|_p)$$

Given $\psi \in \mathcal{S}(\mathbb{R}^d)$ (or $\mathcal{S}_p(\mathbb{R}^d)$) and $\phi \in \mathcal{S}'(\mathbb{R}^d)$ (or $\mathcal{S}_{-p}(\mathbb{R}^d)$), the action of ϕ on ψ will be denoted by $\langle \phi, \psi \rangle$.

Let $\{h_n : n \in \mathbb{Z}_+^d\}$ be the Hermite functions, where $\mathbb{Z}_+^d := \{n = (n_1, \dots, n_d) : n_i \text{ non-negative integers}\}$. If $n = (n_1, \dots, n_d)$, we define $|n| := n_1 + \dots + n_d$. Note that $\{h_n^p : n \in \mathbb{Z}_+^d\}$ forms an orthonormal basis for $\mathcal{S}_p(\mathbb{R}^d)$, where $h_n^p := (2|n| + d)^{-p} h_n$.

Consider the derivative maps denoted by $\partial_i : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$ for $i = 1, \dots, d$. We can extend these maps by duality to $\partial_i : \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$ as follows: for $\psi \in \mathcal{S}'(\mathbb{R}^d)$,

$$\langle \partial_i \psi, \phi \rangle := -\langle \psi, \partial_i \phi \rangle, \quad \forall \phi \in \mathcal{S}(\mathbb{R}^d).$$

Let $\{e_i : i = 1, \dots, d\}$ be the standard basis vectors in \mathbb{R}^d . Then for any $n = (n_1, \dots, n_d) \in \mathbb{Z}_+^d$ we have (see [3, Appendix A.5])

$$\partial_i h_n = \sqrt{\frac{n_i}{2}} h_{n-e_i} - \sqrt{\frac{n_i+1}{2}} h_{n+e_i},$$

with the convention that for a multi-index $n = (n_1, \dots, n_d)$, if $n_i < 0$ for some i , then $h_n \equiv 0$. Above recurrence implies that $\partial_i : \mathcal{S}_p(\mathbb{R}^d) \rightarrow \mathcal{S}_{p-\frac{1}{2}}(\mathbb{R}^d)$ is a bounded linear operator.

For $x \in \mathbb{R}^d$, let τ_x denote the translation operators on $\mathcal{S}(\mathbb{R}^d)$ defined by $(\tau_x \phi)(y) := \phi(y - x), \forall y \in \mathbb{R}^d$. This operators can be extended to $\tau_x : \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$ by

$$\langle \tau_x \phi, \psi \rangle := \langle \phi, \tau_{-x} \psi \rangle, \quad \forall \psi \in \mathcal{S}(\mathbb{R}^d).$$

Proposition 2.1. *The translation operators $\tau_x, x \in \mathbb{R}^d$ have the following properties:*

- (a) *For $x \in \mathbb{R}^d$ and any $p \in \mathbb{R}$, $\tau_x : \mathcal{S}_p(\mathbb{R}^d) \rightarrow \mathcal{S}_p(\mathbb{R}^d)$ is a bounded linear map. In particular, there exists a real polynomial P_k of degree $k = 2(\lfloor p \rfloor + 1)$ such that*

$$\|\tau_x \phi\|_p \leq P_k(|x|) \|\phi\|_p, \forall \phi \in \mathcal{S}_p(\mathbb{R}^d).$$

- (b) *For any $x \in \mathbb{R}^d$ and any $i = 1, \dots, d$ we have*

$$\tau_x \partial_i = \partial_i \tau_x.$$

Proof. See [16, Theorem 2.1] for the proof of part (a). We prove part (b).

Fix an element $\psi \in \mathcal{S}(\mathbb{R}^d)$. Then for $y \in \mathbb{R}^d$,

$$(\tau_x \partial_i \psi)(y) = (\partial_i \psi)(y - x) = \partial_i(\psi(y - x)) = (\partial_i \tau_x \psi)(y)$$

i.e. $\tau_x \partial_i \psi = \partial_i \tau_x \psi$. Via duality we can prove $\partial_i \tau_x \phi = \tau_x \partial_i \phi$ for all $\phi \in \mathcal{S}'(\mathbb{R}^d)$. \square

3. STOCHASTIC INTEGRALS

In this section, we review basic properties of stochastic integrals, specifically those with Hermite-Sobolev valued integrands. Unless stated otherwise we shall use the following notations throughout this section. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ be a filtered complete probability space satisfying the usual conditions. For any real valued martingale M or a process of finite variation A or a semimartingale X , we assume $M_0 \equiv 0, A_0 \equiv 0, X_0 \equiv 0$. Unless stated otherwise stopping times or adapted processes will be with respect to the filtration (\mathcal{F}_t) and any such real valued process (martingales, processes of finite variation or semimartingales) will be assumed to have rcll (right continuous with left limits) paths. For our purpose, we do not require the full generality of stochastic integration on Hilbert spaces as given in [9, Chapter 4 and 5].

3.1. Stochastic integral with respect to a real valued local \mathcal{L}^2 martingale.

Let $\{M_t\}$ be a real valued (\mathcal{F}_t) adapted local \mathcal{L}^2 martingale with rcll paths and $M_0 = 0$. Let $\{\langle M \rangle_t\}$ denote the predictable quadratic variation of M .

Proposition 3.1. *Let $p \in \mathbb{R}$. Let $\{G_t\}$ be an $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued predictable process such that there exists a localizing sequence $\{\tau_n\}$ with the following property: for all $t > 0$ and all positive integers n ,*

$$\mathbb{E} \int_0^{t \wedge \tau_n} \|G_s\|_{-p}^2 d\langle M \rangle_s < \infty.$$

Then $\{\int_0^t G_s dM_s\}$ is an $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued $\{\mathcal{F}_t\}$ adapted local \mathcal{L}^2 martingale.

Let \mathbb{K} be a real separable Hilbert space and $T : \mathcal{S}_{-p}(\mathbb{R}^d) \rightarrow \mathbb{K}$ be a bounded linear operator. Then a.s. $t \geq 0$,

$$T \int_0^t G_s dM_s = \int_0^t T G_s dM_s.$$

In particular, for any $\phi \in \mathcal{S}(\mathbb{R}^d)$, a.s. for all $t \geq 0$

$$(3.1) \quad \left\langle \int_0^t G_s dM_s, \phi \right\rangle = \int_0^t \langle G_s, \phi \rangle dM_s.$$

Proof. For simplicity, assume that M is an \mathcal{L}^2 martingale and $\tau_n = \infty, \forall n$. If G is a predictable step process of the form: $G := \sum_{i=1}^n \mathbb{1}_{(t_{i-1}, t_i]} g_i$ where n is a positive integer, t_0, t_1, \dots, t_n are real numbers satisfying $0 \leq t_0 < t_1 < \dots < t_n$ and g_i are an $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued, $\mathcal{F}_{t_{i-1}}$ measurable random variable. Then

$$\int_0^t G_s dM_s := \sum_{i=1}^n (M_{t \wedge t_i} - M_{t \wedge t_{i-1}}) g_i$$

and

$$\begin{aligned} \mathbb{E} \left\| \int_0^t G_s dM_s \right\|_{-p}^2 &= \mathbb{E} \sum_{i=1}^n \|g_i\|_{-p}^2 (M_{t \wedge t_i} - M_{t \wedge t_{i-1}})^2 \\ &= \mathbb{E} \sum_{i=1}^n \|g_i\|_{-p}^2 (\langle M \rangle_{t \wedge t_i} - \langle M \rangle_{t \wedge t_{i-1}}) \\ &= \mathbb{E} \int_0^t \|G_s\|_{-p}^2 d\langle M \rangle_s. \end{aligned}$$

In view of the above isometry we can extend the stochastic integral to predictable processes $\{G_t\}$ satisfying the integrability condition as mentioned in the statement. Proofs of (\mathcal{F}_t) adaptedness and rcl paths are standard.

If $T : \mathcal{S}_{-p}(\mathbb{R}^d) \rightarrow \mathbb{K}$ is a bounded linear operator, then a.s. $t \geq 0$,

$$T \int_0^t G_s dM_s = \int_0^t T G_s dM_s$$

holds for predictable step processes. The relation then extends to all G satisfying the integrability condition. \square

Remark 3.2. (1) If the martingale is continuous, then we can define the integrals for integrands $\{G_t\}$ which are progressively measurable.
 (2) For continuous processes, equation (3.1) was pointed out in [14, Proposition 1.3(a)].

3.2. Stochastic Integral with respect to a real finite variation process. Let $\{A_t\}$ be a real valued (\mathcal{F}_t) -adapted process of finite variation with right continuous paths. We denote its total variation process by $\{V_{[0,t]}(A)\}$.

Let $\{G_t\}$ be an $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued norm-bounded (i.e. there exists a constant $R > 0$ such that a.s. $\|G_t\|_{-p} \leq R$ for all t) predictable process.

Observe that for all $t \geq 0$ and all ω

$$(3.2) \quad \int_0^t \|G_s\|_{-p} |dA_s| \leq R \cdot V_{[0,t]}(A) < \infty,$$

which allows us to define the $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued random variable $\int_0^t G_s dA_s$ as a Bochner integral. Note that the process $\{\int_0^t G_s dA_s\}$ is of finite variation and has rcl paths. Furthermore, for each $\phi \in \mathcal{S}(\mathbb{R}^d)$, $t \geq 0$ we have

$$(3.3) \quad \left\langle \int_0^t G_s dA_s, \phi \right\rangle = \int_0^t \langle G_s, \phi \rangle dA_s.$$

For continuous processes, this result was pointed out in [14, Proposition 1.3(a)].

3.3. Stochastic Integral with respect to a real semimartingale. We recall the decomposition of local martingales ([6, Lemma 23.5], [12, Chapter III, Theorem 25]).

Theorem 3.3 (Decomposition of Local Martingales). *Given a local martingale $\{M_t\}$, there exist two local martingales $\{M'_t\}$, $\{M''_t\}$ one of which has bounded jumps and the other is of locally integrable variation and a.s.*

$$M_t = M'_t + M''_t, \forall t \geq 0.$$

Since any local martingale with bounded jumps is locally \mathcal{L}^2 , any real semimartingale X has a decomposition (not necessarily unique), a.s. $X_t = M_t + A_t, t \geq 0$, where $\{M_t\}$ is a local \mathcal{L}^2 martingale and $\{A_t\}$ is a process of finite variation. Let $\{G_t\}$ be an $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued norm-bounded (i.e. there exists a constant $R > 0$ such that a.s. $\|G_t\|_{-p} \leq R$ for all t) predictable process. Now the stochastic integral of $\{G_t\}$ with respect to $\{X_t\}$ is defined to be:

$$\int_0^t G_s dX_s := \int_0^t G_s dM_s + \int_0^t G_s dA_s, t \geq 0.$$

Theorem 3.4. *The process $\{\int_0^t G_s dX_s\}$ is well-defined, i.e. the definition does not depend on the decomposition $X = M + A$.*

To prove this, we first recall the following result.

Lemma 3.5. *Let $\{V_t\}$ be a real valued bounded predictable processes. Let $\{M_t\}$ be an \mathcal{L}^2 martingale and $\{A_t\}$ be a process of finite variation such that a.s.*

$$M_t = A_t, \forall t \geq 0.$$

Then a.s.

$$\int_0^t V_s dM_s = \int_0^t V_s dA_s, \forall t \geq 0.$$

Proof. This result is included in the proof of Theorem 23.4 in [6]. \square

Proof of Theorem 3.4. First assume that M is an \mathcal{L}^2 martingale. By Lemma 3.5, for each $\phi \in \mathcal{S}(\mathbb{R}^d)$, the process $\{\int_0^t \langle G_s, \phi \rangle dM_s + \int_0^t \langle G_s, \phi \rangle dA_s\}$ does not depend on the decomposition $X = M + A$. Now varying ϕ in the countable set $\{h_n : n \in \mathbb{Z}_+^d\}$, we get a common null set $\tilde{\Omega}$ such that for all $\omega \in \Omega \setminus \tilde{\Omega}$, for all $n \in \mathbb{Z}_+^d$ and for all $t \geq 0$, we have

$$\left\langle \int_0^t G_s dM_s + \int_0^t G_s dA_s, h_n \right\rangle = \int_0^t \langle G_s, h_n \rangle dM_s + \int_0^t \langle G_s, h_n \rangle dA_s.$$

This identifies the $\mathcal{S}_{-p}(\mathbb{R}^d)$ process $\{\int_0^t G_s dM_s + \int_0^t G_s dA_s\}$ independent of the decomposition $X = M + A$.

If M is a local \mathcal{L}^2 martingale, the proof can be completed using stopping time arguments. \square

4. THE ITÔ FORMULA

Given $\phi \in \mathcal{S}'(\mathbb{R}^d)$, there exists a $p > 0$ such that $\phi \in \mathcal{S}_{-p}(\mathbb{R}^d)$. Let $X_t = (X_t^1, \dots, X_t^d)$ be an \mathbb{R}^d valued (\mathcal{F}_t) semimartingale with rcll paths with the decomposition a.s.

$$X_t = X_0 + M_t + A_t, t \geq 0$$

where $M_t = (M_t^1, \dots, M_t^d)$ is an \mathbb{R}^d valued locally square integrable martingale and $A_t = (A_t^1, \dots, A_t^d)$ is an \mathbb{R}^d valued process of finite variation. Both $\{M_t\}$ and $\{A_t\}$ have rcll paths and $M_0 = 0 = A_0$ a.s. By Proposition 2.1, $\{\tau_{X_t}\phi\}$ is an $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued process. Recall that the process $\{X_{t-}\}$ defined by

$$X_{t-} := \begin{cases} X_0, & \text{if } t = 0. \\ \lim_{s \downarrow t} X_s, & \text{if } t > 0. \end{cases},$$

is predictable (see [5, Chapter I, 2.6 Proposition]).

Lemma 4.1. *Let $\phi, \{X_t\}$ be as above. Then for any $1 \leq i \leq d$ and $1 \leq j \leq d$,*

- (1) *$\{\tau_{X_{t-}}\phi\}$ is an $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued predictable process.*
- (2) *$\{\partial_i \tau_{X_{t-}}\phi\}$ is an $\mathcal{S}_{-p-\frac{1}{2}}(\mathbb{R}^d)$ valued predictable process.*
- (3) *$\{\partial_{ij}^2 \tau_{X_{t-}}\phi\}$ is an $\mathcal{S}_{-p-1}(\mathbb{R}^d)$ valued predictable process.*

Proof. Since $\{X_{t-}\}$ is predictable and $x \mapsto \tau_x \phi : \mathbb{R}^d \rightarrow \mathcal{S}_{-p}(\mathbb{R}^d)$ is continuous (see the proof of [17, Proposition 3.1]), the process $\{\tau_{X_{t-}}\phi\}$ is predictable.

For any $1 \leq i \leq d$, we have $\tau_x(\partial_i \phi) = \partial_i \tau_x \phi$ (see Proposition 2.1) and $\partial_i : \mathcal{S}_{-p}(\mathbb{R}^d) \rightarrow \mathcal{S}_{-p-\frac{1}{2}}(\mathbb{R}^d)$ is a bounded linear operator. Hence $\{\partial_i \tau_{X_{t-}}\phi\}$ is an $\mathcal{S}_{-p-\frac{1}{2}}(\mathbb{R}^d)$ valued predictable process.

Similarly for $1 \leq i, j \leq d$, the processes $\{\partial_{ij}^2 \tau_{X_{t-}}\phi\}$ are $\mathcal{S}_{-p-1}(\mathbb{R}^d)$ valued predictable processes. \square

Using [9, 25.5 Corollary 3], there exists a set $\tilde{\Omega}$ with $P(\tilde{\Omega}) = 1$ such that

$$\sum_{s \leq t} |\Delta X_s|^2 < \infty, \forall t > 0, \omega \in \tilde{\Omega}.$$

If $\omega \in \tilde{\Omega}$, then there are at most countably many jumps of X on $[0, t]$. The following can be easily established.

Lemma 4.2. *Fix $\omega \in \tilde{\Omega}$.*

- (i) *Fix $t > 0$. Let $\{t_n\}$ be a strictly increasing sequence converging to t . Then*

$$\lim_{n \rightarrow \infty} \sum_{s \leq t_n} |\Delta X_s(\omega)|^2 = \sum_{s < t} |\Delta X_s(\omega)|^2.$$

- (ii) *Fix $t \geq 0$. Let $\{t_n\}$ be a strictly decreasing sequence converging to t . Then*

$$\lim_{n \rightarrow \infty} \sum_{t_n < s \leq t_1} |\Delta X_s(\omega)|^2 = \sum_{t < s \leq t_1} |\Delta X_s(\omega)|^2.$$

Using Lemma 4.2, we get the following estimate which we use later in Theorem 4.5.

Lemma 4.3. *Let $\phi, \{X_t\}$ be as above. Fix $\omega \in \tilde{\Omega}$. Fix $\psi \in \mathcal{S}(\mathbb{R}^d)$. Then for all $s \leq t$*

$$\left| \left\langle \tau_{X_s} \phi - \tau_{X_{s-}} \phi + \sum_{i=1}^d (\Delta X_s^i \partial_i \tau_{X_{s-}} \phi), \psi \right\rangle \right| \leq C(t, \omega) \cdot |\Delta X_s|^2 \|\psi\|_{p+1},$$

and hence

$$(4.1) \quad \|\tau_{X_s} \phi - \tau_{X_{s-}} \phi + \sum_{i=1}^d (\Delta X_s^i \partial_i \tau_{X_{s-}} \phi)\|_{-p-1} \leq C(t, \omega) \cdot |\Delta X_s|^2.$$

Here $C(t, \omega)$ is a positive constant depending on t, ω and is also non-decreasing in t . In particular,

$$\tau_{X_t} \phi - \tau_{X_{t-}} \phi + \sum_{i=1}^d (\Delta X_t^i \partial_i \tau_{X_{t-}} \phi) = 0, \text{ if } |\Delta X_t| = 0.$$

Note 4.4. To simplify notations, we shall write $C(t)$ instead of $C(t, \omega)$.

Proof of Lemma 4.3. By [14, Proposition 1.4], there exists some positive integer n such that the map $x \mapsto \tau_x \phi \in \mathcal{S}_{-n}(\mathbb{R}^d)$ is a C^2 map. For any fixed $\psi \in \mathcal{S}(\mathbb{R}^d)$ we have $x \mapsto \langle \tau_x \phi, \psi \rangle$ is a C^2 map and

$$\begin{aligned} \partial_i \langle \tau_x \phi, \psi \rangle &= \partial_i \langle \phi, \psi(\cdot + x) \rangle = \langle \phi, \partial_i \psi(\cdot + x) \rangle \\ &= \langle \phi, \tau_{-x} \partial_i \psi \rangle = - \langle \partial_i \tau_x \phi, \psi \rangle. \end{aligned}$$

For any $1 \leq i, j \leq d$, we have $\partial_{ij}^2 = \partial_i \partial_j = \partial_j \partial_i$ on $\mathcal{S}'(\mathbb{R}^d)$ and hence $\partial_{ij}^2 : \mathcal{S}_{-p}(\mathbb{R}^d) \rightarrow \mathcal{S}_{-p-1}(\mathbb{R}^d)$ is a bounded linear operator. Then there exists a constant $\alpha > 0$ such that

$$(4.2) \quad \|\partial_{ij}^2 \theta\|_{-p-1} \leq \alpha \|\theta\|_{-p}, \forall \theta \in \mathcal{S}_{-p}(\mathbb{R}^d).$$

We follow the proof of [6, Theorem 23.7] and define $B(t, \omega) := \{x \in \mathbb{R}^d : |x| \leq \sup_{s \leq t} |X_s(\omega)|\}$. Then using Taylor's formula for the C^2 map $x \mapsto \langle \tau_x \phi, \psi \rangle$, we have for all $s \leq t$

$$\begin{aligned} & \left| \left\langle \tau_{X_s} \phi - \tau_{X_{s-}} \phi + \sum_{i=1}^d (\Delta X_s^i \partial_i \tau_{X_{s-}} \phi), \psi \right\rangle \right| \\ &= \left| \langle \tau_{X_s} \phi, \psi \rangle - \langle \tau_{X_{s-}} \phi, \psi \rangle + \sum_{i=1}^d \langle \partial_i \tau_{X_{s-}} \phi, \psi \rangle \Delta X_s^i \right| \\ &= \left| \langle \tau_{X_s} \phi, \psi \rangle - \langle \tau_{X_{s-}} \phi, \psi \rangle - \sum_{i=1}^d \partial_i \langle \tau_{X_{s-}} \phi, \psi \rangle \Delta X_s^i \right| \\ &\leq \frac{1}{2} |\Delta X_s|^2 \left(\sum_{i,j=1}^d \sup_{y \in B(t, \omega)} |\langle \partial_{ij}^2 \tau_y \phi, \psi \rangle| \right) \\ &\leq \frac{1}{2} |\Delta X_s|^2 \left(\sum_{i,j=1}^d \sup_{y \in B(t, \omega)} \|\partial_{ij}^2 \tau_y \phi\|_{-p-1} \right) \|\psi\|_{p+1} \\ &\leq \frac{\alpha}{2} |\Delta X_s|^2 \left(\sup_{y \in B(t, \omega)} \|\tau_y \phi\|_{-p} \right) \|\psi\|_{p+1} \text{ (using (4.2)).} \end{aligned}$$

Define $C(t, \omega) := \frac{\alpha}{2} \left(\sup_{y \in B(t, \omega)} \|\tau_y \phi\|_{-p} \right)$. Then $C(t, \omega)$ is non-decreasing in t and for all $s \leq t$

$$\left| \left\langle \tau_{X_s} \phi - \tau_{X_{s-}} \phi + \sum_{i=1}^d (\Delta X_s^i \partial_i \tau_{X_{s-}} \phi), \psi \right\rangle \right| \leq C(t, \omega) \cdot |\Delta X_s|^2 \|\psi\|_{p+1}.$$

From above estimate we have

$$\|\tau_{X_s} \phi - \tau_{X_{s-}} \phi + \sum_{i=1}^d (\Delta X_s^i \partial_i \tau_{X_{s-}} \phi)\|_{-p-1} \leq C(t, \omega) \cdot |\Delta X_s|^2.$$

In particular $\tau_{X_t}\phi - \tau_{X_{t-}}\phi + \sum_{i=1}^d(\Delta X_t^i \partial_i \tau_{X_{t-}}\phi) = 0$ if $|\Delta X_t| = 0$. \square

For any $i, j = 1, \dots, d$, let $\{[X^i, X^j]_t^c\}$ denote the continuous part of $\{[X^i, X^j]_t\}$. We now prove the main result of this paper.

Theorem 4.5. *Let $p > 0$ and $\phi \in \mathcal{S}_{-p}(\mathbb{R}^d)$. Let $X = (X^1, \dots, X^d)$ be a \mathbb{R}^d valued (\mathcal{F}_t) semimartingale. Let ΔX_s^i denote the jump of X_s^i . Then $\{\tau_{X_t}\phi\}$ is an $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued semimartingale and*

$$\sum_{s \leq t} \left[\tau_{X_s}\phi - \tau_{X_{s-}}\phi + \sum_{i=1}^d (\Delta X_s^i \partial_i \tau_{X_{s-}}\phi) \right]$$

is a $\mathcal{S}_{-p-1}(\mathbb{R}^d)$ valued process of finite variation and we have the following equality in $\mathcal{S}_{-p-1}(\mathbb{R}^d)$, a.s.

$$(4.3) \quad \begin{aligned} \tau_{X_t}\phi &= \tau_{X_0}\phi - \sum_{i=1}^d \int_0^t \partial_i \tau_{X_{s-}}\phi dX_s^i + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \partial_{ij}^2 \tau_{X_{s-}}\phi d[X^i, X^j]_s^c \\ &\quad + \sum_{s \leq t} \left[\tau_{X_s}\phi - \tau_{X_{s-}}\phi + \sum_{i=1}^d (\Delta X_s^i \partial_i \tau_{X_{s-}}\phi) \right], \quad t \geq 0. \end{aligned}$$

Proof. We proceed in steps.

Step 1: Let $\tilde{\Omega}$ be as in Lemma 4.3. Then $\omega \in \tilde{\Omega}$ implies (see equation (4.1))

$$(4.4) \quad \sum_{s \leq t} \left\| \tau_{X_s}\phi - \tau_{X_{s-}}\phi + \sum_{i=1}^d (\Delta X_s^i \partial_i \tau_{X_{s-}}\phi) \right\|_{-p-1} \leq C(t) \sum_{s \leq t} |\Delta X_s|^2 < \infty.$$

Recall that if $\omega \in \tilde{\Omega}$, then there are at most countably many jumps of X on $[0, t]$. In view of the above estimate we define for any $t \geq 0$

$$Y_t(\omega) := \sum_{s \leq t} \left[\tau_{X_s(\omega)}\phi - \tau_{X_{s-}(\omega)}\phi + \sum_{i=1}^d (\Delta X_s^i(\omega) \partial_i \tau_{X_{s-}(\omega)}\phi) \right], \quad \omega \in \tilde{\Omega}$$

and set $Y_t(\omega) := 0$, $\omega \in (\tilde{\Omega})^c$. Then $\{Y_t\}$ is a well-defined $\mathcal{S}_{-p-1}(\mathbb{R}^d)$ valued (\mathcal{F}_t) adapted process.

Step 2: Now we show $\{Y_t\}$ has rcl paths and is a process of finite variation. Fix $\omega \in \tilde{\Omega}$. We claim

$$(i) \quad Y_{t-} = \sum_{s < t} \left[\tau_{X_s}\phi - \tau_{X_{s-}}\phi + \sum_{i=1}^d (\Delta X_s^i \partial_i \tau_{X_{s-}}\phi) \right], \quad t > 0.$$

$$(ii) \quad Y_{t+} = \sum_{s \leq t} \left[\tau_{X_s}\phi - \tau_{X_{s-}}\phi + \sum_{i=1}^d (\Delta X_s^i \partial_i \tau_{X_{s-}}\phi) \right] = Y_t, \quad t \geq 0.$$

We prove (i). Let $\{t_m\}$ be an increasing sequence converging to t . Then

$$\begin{aligned} & \left\| \sum_{s < t} \left[\tau_{X_s}\phi - \tau_{X_{s-}}\phi + \sum_{i=1}^d (\Delta X_s^i \partial_i \tau_{X_{s-}}\phi) \right] - Y_{t_m} \right\|_{-p-1} \\ &= \left\| \sum_{t_m < s < t} \left[\tau_{X_s}\phi - \tau_{X_{s-}}\phi + \sum_{i=1}^d (\Delta X_s^i \partial_i \tau_{X_{s-}}\phi) \right] \right\|_{-p-1} \\ &\leq \sum_{t_m < s < t} \left\| \tau_{X_s}\phi - \tau_{X_{s-}}\phi + \sum_{i=1}^d (\Delta X_s^i \partial_i \tau_{X_{s-}}\phi) \right\|_{-p-1} \end{aligned}$$

$$\begin{aligned}
&\leq C(t) \sum_{t_m < s < t} |\Delta X_s|^2 \text{ (using (4.1))} \\
&= C(t) \left[\sum_{s < t} |\Delta X_s|^2 - \sum_{s \leq t_m} |\Delta X_s|^2 \right] \xrightarrow{m \rightarrow \infty} 0 \text{ (by Lemma 4.2(i)).}
\end{aligned}$$

This proves (i). Proof of (ii) is similar. Now using (i),(ii) we have on $\tilde{\Omega}$

$$\Delta Y_t = \tau_{X_t} \phi - \tau_{X_{t-}} \phi + \sum_{i=1}^d (\Delta X_t^i \partial_i \tau_{X_{t-}} \phi),$$

and $\Delta Y_t = 0$ if $\Delta X_t = 0$. Now using (4.1), we also have

$$\sum_{s \leq t} \|\Delta Y_s\|_{-p-1} \leq C(t) \sum_{s \leq t} |\Delta X_s|^2 < \infty, \quad \omega \in \tilde{\Omega}$$

and $Y_t = \sum_{s \leq t} \Delta Y_s$. We have shown $\{Y_t\}$ has rcll paths. Now we show that $\{Y_t\}$ has paths of finite variation.

Let $\omega \in \tilde{\Omega}$ and $t > 0$. Let $\mathbb{P} = \{0 = t_0 < t_1 < \dots < t_m = t\}$ be a partition of $[0, t]$. Then

$$\begin{aligned}
&\sum_{i=1}^m \|Y_{t_i} - Y_{t_{i-1}}\|_{-p-1} \\
&= \sum_{i=1}^m \left\| \sum_{t_{i-1} < s \leq t_i} \left[\tau_{X_s} \phi - \tau_{X_{s-}} \phi + \sum_{i=1}^d (\Delta X_s^i \partial_i \tau_{X_{s-}} \phi) \right] \right\|_{-p-1} \\
&\leq \sum_{i=1}^m \sum_{t_{i-1} < s \leq t_i} \left\| \tau_{X_s} \phi - \tau_{X_{s-}} \phi + \sum_{i=1}^d (\Delta X_s^i \partial_i \tau_{X_{s-}} \phi) \right\|_{-p-1} \\
&= \sum_{s \leq t} \left\| \tau_{X_s} \phi - \tau_{X_{s-}} \phi + \sum_{i=1}^d (\Delta X_s^i \partial_i \tau_{X_{s-}} \phi) \right\|_{-p-1} \\
&\leq C(t) \sum_{s \leq t} |\Delta X_s|^2.
\end{aligned}$$

Since the quantity $C(t) \sum_{s \leq t} |\Delta X_s|^2$ is independent of the choice of the partition \mathbb{P} , we have $\{Y_t\}$ is of finite variation with

$$Var_{[0,t]}(Y) \leq C(t) \sum_{s \leq t} |\Delta X_s|^2$$

on $\tilde{\Omega}$.

Step 3: To complete the proof we need to verify the following equality in $\mathcal{S}_{-p-1}(\mathbb{R}^d)$, a.s. for all $t \geq 0$

$$Y_t = \tau_{X_t} \phi - \tau_{X_0} \phi + \sum_{i=1}^d \int_0^t \partial_i \tau_{X_{s-}} \phi dX_s^i - \frac{1}{2} \sum_{i,j=1}^d \int_0^t \partial_{ij}^2 \tau_{X_{s-}} \phi d[X^i, X^j]_s^c.$$

First we assume that the processes $\{X_{t-}\}, \{[X^i, X^j]_t^c\}, i, j = 1, \dots, d$ are bounded. Since $\partial_i : \mathcal{S}_{-p}(\mathbb{R}^d) \rightarrow \mathcal{S}_{-p-\frac{1}{2}}(\mathbb{R}^d)$ is a bounded linear operator,

by Proposition 2.1, we have for all $t \geq 0, i = 1, \dots, d$

$$\|\partial_i \tau_{X_{t-}} \phi\|_{-p-\frac{1}{2}} \leq C \|\tau_{X_{t-}} \phi\|_{-p} \leq C P_k(|X_{t-}|) \|\phi\|_{-p} \leq C',$$

where $C, C' > 0$ are appropriate constants. Similarly, there exists a constant $C'' > 0$ such that

$$\|\partial_{ij} \tau_{X_{t-}} \phi\|_{-p-1} \leq C'', \forall t \geq 0, i, j = 1, \dots, d.$$

Hence $\{\tau_{X_{t-}} \phi\}, \{\partial_i \tau_{X_{t-}} \phi\}, \{\partial_{ij}^2 \tau_{X_{t-}} \phi\}$ are norm-bounded predictable processes (see Lemma 4.1). As per the results mentioned in the previous section, we can define stochastic integrals

$$I_t^1 := \sum_{i=1}^d \int_0^t \partial_i \tau_{X_{s-}} \phi dX_s^i, \quad I_t^2 := \sum_{i,j=1}^d \int_0^t \partial_{ij}^2 \tau_{X_{s-}} \phi d[X^i, X^j]_s^c, \quad t \geq 0$$

which are respectively $\mathcal{S}_{-p-\frac{1}{2}}(\mathbb{R}^d)$ and $\mathcal{S}_{-p-1}(\mathbb{R}^d)$ valued and have rcll paths.

For $n \in \mathbb{Z}_+^d$ applying the Itô formula (see [6, Theorem 23.7]) to the C^2 map $x \mapsto \langle \tau_x \phi, h_n \rangle$ we have, a.s. for all $t \geq 0$

$$\begin{aligned} \langle \tau_{X_t} \phi, h_n \rangle &= \langle \tau_{X_0} \phi, h_n \rangle - \underbrace{\sum_{i=1}^d \int_0^t \langle \partial_i \tau_{X_{s-}} \phi, h_n \rangle dX_s^i}_{= \langle I_t^1, h_n \rangle} \\ &+ \frac{1}{2} \underbrace{\sum_{i,j=1}^d \int_0^t \langle \partial_{ij}^2 \tau_{X_{s-}} \phi, h_n \rangle d[X^i, X^j]_s^c}_{= \langle I_t^2, h_n \rangle} \\ &+ \sum_{s \leq t} \left[\langle \tau_{X_s} \phi, h_n \rangle - \langle \tau_{X_{s-}} \phi, h_n \rangle + \sum_{i=1}^d \langle \partial_i \tau_{X_{s-}} \phi, h_n \rangle \Delta X_s^i \right], \end{aligned} \quad (4.5)$$

where ΔX_s^i denotes the jump of X_s^i . Now varying n in the countable set \mathbb{Z}_+^d , we get a common null set $\tilde{\Omega}$ such that for all $\omega \in \Omega \setminus \tilde{\Omega}$, for all $n \in \mathbb{Z}_+^d$ and for all $t \geq 0$, we have

$$\begin{aligned} \langle (\tau_{X_t} \phi - \tau_{X_0} \phi) + \sum_{i=1}^d \int_0^t \partial_i \tau_{X_{s-}} \phi dX_s^i \\ - \frac{1}{2} \sum_{i,j=1}^d \int_0^t \partial_{ij}^2 \tau_{X_{s-}} \phi d[X^i, X^j]_s^c - Y_t, h_n \rangle = 0. \end{aligned}$$

Recall that $\{h_n^q : n \in \mathbb{Z}_+^d\}$ is an orthonormal basis for $\mathcal{S}_{-q}(\mathbb{R}^d)$, where $h_n^q = (2k+d)^{-q}$ with $k = |n| = n_1 + \dots + n_d$. From the previous relation, we get the required equality in $\mathcal{S}_{-p-1}(\mathbb{R}^d)$ for semimartingales $\{X_t\}$ such that $\{X_{t-}\}, \{[X^i, X^j]_t^c\}, i, j = 1, \dots, d$ are bounded.

Step 4: Now suppose at least one of $\{X_{t-}\}, \{[X^i, X^j]_t^c\}, i, j = 1, \dots, d$ is not bounded. Then define

$$\bar{\sigma}_n := \inf\{t \geq 0 : |[X^i, X^j]_t^c| \geq n, i, j = 1, \dots, d\}$$

and

$$\tilde{\sigma}_n := \inf\{t \geq 0 : |X_t| \geq n\},$$

where $|\cdot|$ represents the Euclidean norms in the appropriate space \mathbb{R}^m ($m = 1$ or d). Set $\sigma_n = \bar{\sigma}_n \wedge \tilde{\sigma}_n$. Then $\{([X^i, X^j]^c)_t^{\sigma_n}\}$, $i, j = 1, \dots, d$ are bounded.

If $|X_0(\omega)| > n$ for some w , then $\tau_n(\omega) = 0$. Such ω does not contribute to $\sum_{i=1}^d \int_0^{t \wedge \sigma_n} \|\partial_i \tau_{X_{s-}} \xi\|_{p-\frac{1}{2}}^2 d\langle M^i \rangle_s$ etc. So we may assume the processes $\{X_{t-}^{\sigma_n}\}$ are bounded. Hence a.s. in $\mathcal{S}_{-p-1}(\mathbb{R}^d)$ we have for all $t \geq 0$

$$\begin{aligned} \tau_{X_{t \wedge \sigma_n}} \phi &= \tau_{X_0} \phi + \sum_{i=1}^d \int_0^{t \wedge \sigma_n} \partial_i \tau_{X_{s-}} \phi dX_s^i \\ &\quad - \frac{1}{2} \sum_{i,j=1}^d \int_0^{t \wedge \sigma_n} \partial_{ij}^2 \tau_{X_{s-}} \phi d[X^i, X^j]_s^c - Y_t. \end{aligned}$$

Letting n go to infinity we get the result. \square

Given a real valued semimartingale $\{X_t\}$, consider the local time process denoted by $\{L_t(x)\}_{t \in [0, \infty), x \in \mathbb{R}}$. Note that this process is jointly measurable in (x, t, ω) and for each $x \in \mathbb{R}$, $\{L_t(x)\}$ is a continuous adapted process. By the occupation density formula [12, p. 216, Corollary 1], we have for any $\phi \in \mathcal{S}$, a.s.

$$(4.6) \quad \int_{-\infty}^{\infty} L_t(x) \phi(x) dx = \int_0^t \phi(X_{s-}) d[X]_s^c,$$

where $[X]$ stands for $[X, X]$ and $[X]^c$ denotes the continuous part of $[X]$ (also see [13, Proposition 4]). By [12, p. 216, Corollary 2] a.s.

$$\int_{-\infty}^{\infty} L_t(x) dx = \int_0^t d[X]_s^c,$$

which shows a.s. for all t , the map $x \mapsto L_t(x)$ is integrable. We now identify the local time process in \mathcal{S}' .

Proposition 4.6. *The \mathcal{S}' valued process $\{\int_0^t \delta_{X_{s-}} d[X]_s^c\}$ is \mathcal{S}_{-p} valued for any $p > \frac{1}{4}$ and for each t , $\int_0^t \delta_{X_{s-}} d[X]_s^c$ is given by the integrable function $x \mapsto L_t(x)$.*

Proof. Note that for any fixed $x \in \mathbb{R}$, the distribution δ_x is in \mathcal{S}_{-p} for any $p > \frac{1}{4}$ and furthermore for such a p we have $\sup_{x \in \mathbb{R}^d} \|\delta_x\|_{-p} < \infty$ (see [17, Theorem 4.1]). Also $\tau_x \delta_0 = \delta_x, \forall x \in \mathbb{R}^d$. Hence $\{\delta_{X_{t-}}\}$ is an \mathcal{S}_{-p} valued norm-bounded predictable process (see Lemma 4.1). Then we can define the \mathcal{S}_{-p} valued process $\{\int_0^t \delta_{X_{s-}} d[X]_s^c\}$ for any $p > \frac{1}{4}$. But for any integer $n \geq 0$, by (4.6) a.s. for all $t \geq 0$

$$\begin{aligned} \left\langle \int_0^t \delta_{X_{s-}} d[X]_s^c, h_n \right\rangle &= \int_0^t \langle \delta_{X_{s-}}, h_n \rangle d[X]_s^c \\ &= \int_0^t h_n(X_{s-}) d[X]_s^c = \int_{-\infty}^{\infty} L_t(x) h_n(x) dx \end{aligned}$$

Then there exists a P null set $\tilde{\Omega}$ such that on $\Omega \setminus \tilde{\Omega}$ for all integers $n \geq 0$ and all $t \geq 0$

$$\left\langle \int_0^t \delta_{X_{s-}} d[X]_s^c, h_n \right\rangle = \int_{-\infty}^{\infty} L_t(x) h_n(x) dx.$$

Since $\{(2n+1)^p h_n : n = 0, 1, \dots\}$ is an orthonormal basis for \mathcal{S}_{-p} , for each t , the \mathcal{S}' valued random variable $\int_0^t \delta_{X_{s-}} d[X]_s^c$ is given by the function $x \mapsto L_t(x)$. \square

We now apply Theorem 4.5 to a Lévy process to obtain the existence of solutions of certain classes of stochastic differential equations in the Hermite-Sobolev spaces. This is similar in spirit to the same obtained in [15, Theorem 3.4 and Lemma 3.6] for continuous processes.

Let $p \in \mathbb{R}$. Let $\phi \in \mathcal{S}_p$ and $\sigma, b \in \mathcal{S}_{-p}$. Let $F, G : \mathcal{S}_p \times \mathbb{R} \rightarrow \mathbb{R}$ and let $\bar{F}, \bar{G} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be given by $\bar{F}(x, \tilde{x}) := F(\tau_x \phi, \tilde{x})$, $\bar{G}(x, \tilde{x}) := G(\tau_x \phi, \tilde{x})$. Let $\{B_t\}$ be the standard (\mathcal{F}_t) Brownian motion and let N be a Poisson process driven by a Lévy measure ν . Let \tilde{N} denote the compensated measure. Assume that B and N are independent. Let the one-dimensional process $\{X_t\}$ satisfy the following equation: a.s. $t \geq 0$

$$(4.7) \quad \begin{aligned} X_t = & \int_0^t \bar{b}(X_{s-}) ds + \int_0^t \bar{\sigma}(X_{s-}) dB_s \\ & + \int_0^t \int_{(0 < |x| < 1)} \bar{F}(X_{s-}, x) \tilde{N}(dsdx) \\ & + \int_0^t \int_{(|x| \geq 1)} \bar{G}(X_{s-}, x) N(dsdx), \end{aligned}$$

where

- (1) $\bar{\sigma}(x) := \langle \sigma, \tau_x \phi \rangle$, $\bar{b}(x) := \langle b, \tau_x \phi \rangle$ are Lipschitz continuous functions,
- (2) the coefficients \bar{F}, \bar{G} satisfy conditions of [1, Chapter 6, Section 2] with $c = 1$. This parameter c separates the small and large jumps. We assume the integrability condition: a.s.

$$\int_0^t \int_{(0 < |x| < 1)} |\bar{F}(X_{s-}, x)|^2 \nu(dx) ds < \infty, \forall t \geq 0.$$

As an application of Theorem 4.5 we get the next result.

Theorem 4.7. *The \mathcal{S}_p valued process Y defined by $Y_t := \tau_{X_t} \phi$ solves the following stochastic differential equation with equality in \mathcal{S}_{p-1} :*

$$(4.8) \quad \begin{aligned} Y_t(\phi) = & \phi + \int_0^t A(Y_{s-}(\phi)) dB_s + \int_0^t L(Y_{s-}(\phi)) ds \\ & + \int_0^t \int_{(0 < |x| < 1)} (\tau_{F(Y_{s-}(\phi), x)} - Id + F(Y_{s-}(\phi), x) \partial) Y_{s-}(\phi) \nu(dx) ds \\ & + \int_0^t \int_{(0 < |x| < 1)} (\tau_{F(Y_{s-}(\phi), x)} - Id) Y_{s-}(\phi) \tilde{N}(dsdx) \\ & + \int_0^t \int_{(|x| \geq 1)} (\tau_{G(Y_{s-}(\phi), x)} - Id) Y_{s-}(\phi) N(dsdx), \end{aligned}$$

where the operators A, L on \mathcal{S}_p are as follows:

$$A\phi := -\langle \sigma, \phi \rangle \partial \phi,$$

and

$$L\phi := \frac{1}{2} \langle \sigma, \phi \rangle^2 \partial^2 \phi - \langle b, \phi \rangle \partial \phi.$$

Proof. Observe that

$$(4.9) \quad \Delta X_t = \bar{F}(X_{t-}, \Delta X_t) \mathbb{1}_{(0 < |\Delta X_t| < 1)} + \bar{G}(X_{t-}, \Delta X_t) \mathbb{1}_{(|\Delta X_t| \geq 1)}.$$

From (4.9) we make two observations. Firstly, $|\bar{F}(X_{t-}, \Delta X_t)| \mathbb{1}_{(0 < |\Delta X_t| < 1)} \leq 1$. In particular, this implies

$$|\bar{F}(X_{t-}, \Delta X_t)|^4 \mathbb{1}_{(0 < |\Delta X_t| < 1)} \leq |\bar{F}(X_{t-}, \Delta X_t)|^2 \mathbb{1}_{(0 < |\Delta X_t| < 1)}.$$

Secondly, we have the following simplification.

$$\begin{aligned} & \tau_{X_s} \phi - \tau_{X_{s-}} \phi + \Delta X_s \partial \tau_{X_{s-}} \phi \\ &= (\tau_{\Delta X_s} - Id) \tau_{X_{s-}} \phi + \Delta X_s \partial \tau_{X_{s-}} \phi \\ &= \mathbb{1}_{(0 < |\Delta X_s| < 1)} \left(\tau_{\bar{F}(X_{s-}, \Delta X_s)} - Id + \bar{F}(X_{s-}, \Delta X_s) \partial \right) \tau_{X_{s-}} \phi \\ &+ \mathbb{1}_{(|\Delta X_s| \geq 1)} \left(\tau_{\bar{G}(X_{s-}, \Delta X_s)} - Id \right) \tau_{X_{s-}} \phi + \mathbb{1}_{(|\Delta X_s| \geq 1)} \bar{G}(X_{s-}, \Delta X_s) \partial \tau_{X_{s-}} \phi. \end{aligned}$$

Using equation (4.1), we have

$$\begin{aligned} & \mathbb{1}_{(0 < |\Delta X_s| < 1)} \left\| \left(\tau_{\bar{F}(X_{s-}, \Delta X_s)} - Id + \bar{F}(X_{s-}, \Delta X_s) \partial \right) \tau_{X_{s-}} \phi \right\|_{-p-1} \\ & \leq C(s) \mathbb{1}_{(0 < |\Delta X_s| < 1)} |\bar{F}(X_{s-}, \Delta X_s)|^2, \end{aligned}$$

where $t \mapsto C(t)$ is a positive non-decreasing function. Then

$$\begin{aligned} & \int_0^t \int_{(0 < |x| < 1)} \left\| \left(\tau_{\bar{F}(X_{s-}, x)} - Id + \bar{F}(X_{s-}, x) \partial \right) \tau_{X_{s-}} \phi \right\|_{-p-1}^2 \nu(dx) ds \\ & \leq \int_0^t C(s)^2 \int_{(0 < |x| < 1)} |\bar{F}(X_{s-}, x)|^4 \nu(dx) ds \\ & \leq C(t)^2 \int_0^t \int_{(0 < |x| < 1)} |\bar{F}(X_{s-}, x)|^2 \nu(dx) ds < \infty \end{aligned}$$

Similarly

$$\begin{aligned} & \int_0^t \int_{(0 < |x| < 1)} \left\| \left(\tau_{\bar{F}(X_{s-}, x)} - Id \right) \tau_{X_{s-}} \phi \right\|_{-p-\frac{1}{2}}^2 \nu(dx) ds \\ & \leq \tilde{C}(t)^2 \int_0^t \int_{(0 < |x| < 1)} |\bar{F}(X_{s-}, x)|^2 \nu(dx) ds < \infty, \end{aligned}$$

where $t \mapsto \tilde{C}(t)$ is some non-decreasing function. Hence

$$\begin{aligned} & \sum_{s \leq t} [\tau_{X_s} \phi - \tau_{X_{s-}} \phi + \Delta X_s \partial \tau_{X_{s-}} \phi] \\ &= \int_0^t \int_{(0 < |x| < 1)} \left(\tau_{\bar{F}(X_{s-}, x)} - Id + \bar{F}(X_{s-}, x) \partial \right) \tau_{X_{s-}} \phi N(ds dx) \\ &+ \int_0^t \int_{(|x| \geq 1)} \left(\tau_{\bar{G}(X_{s-}, x)} - Id \right) \tau_{X_{s-}} \phi N(ds dx) \\ &+ \int_0^t \int_{(|x| \geq 1)} \bar{G}(X_{s-}, x) \partial \tau_{X_{s-}} \phi N(ds dx) \\ &= \int_0^t \int_{(0 < |x| < 1)} \left(\tau_{\bar{F}(X_{s-}, x)} - Id + \bar{F}(X_{s-}, x) \partial \right) \tau_{X_{s-}} \phi \tilde{N}(ds dx) \end{aligned}$$

$$\begin{aligned}
& + \int_0^t \int_{(0 < |x| < 1)} \left(\tau_{\bar{F}(X_{s-}, x)} - Id + \bar{F}(X_{s-}, x) \partial \right) \tau_{X_{s-}} \phi \nu(dx) ds \\
& + \int_0^t \int_{(|x| \geq 1)} \left(\tau_{\bar{G}(X_{s-}, x)} - Id \right) \tau_{X_{s-}} \phi N(dsdx) \\
& + \int_0^t \int_{(|x| \geq 1)} \bar{G}(X_{s-}, x) \partial \tau_{X_{s-}} \phi N(dsdx).
\end{aligned}$$

Now by the Itô formula (Theorem 4.5)

$$\begin{aligned}
\tau_{X_t} \phi & = \tau_{X_0} \phi + \int_0^t A(\tau_{X_{s-}} \phi) dB_s + \int_0^t L(\tau_{X_{s-}} \phi) ds \\
& - \int_0^t \int_{(0 < |x| < 1)} \bar{F}(X_{s-}, x) \partial \tau_{X_{s-}} \phi \tilde{N}(dsdx) \\
& - \int_0^t \int_{(|x| \geq 1)} \bar{G}(X_{s-}, x) \partial \tau_{X_{s-}} \phi N(dsdx) \\
& + \sum_{s \leq t} [\tau_{X_s} \phi - \tau_{X_{s-}} \phi + \triangle X_s \partial \tau_{X_{s-}} \phi] \\
& = \phi + \int_0^t A(\tau_{X_{s-}} \phi) dB_s + \int_0^t L(\tau_{X_{s-}} \phi) ds \\
& + \int_0^t \int_{(0 < |x| < 1)} \left(\tau_{\bar{F}(X_{s-}, x)} - Id + \bar{F}(X_{s-}, x) \partial \right) \tau_{X_{s-}} \phi \nu(dx) ds \\
& + \int_0^t \int_{(0 < |x| < 1)} \left(\tau_{\bar{F}(X_{s-}, x)} - Id \right) \tau_{X_{s-}} \phi \tilde{N}(dsdx) \\
& + \int_0^t \int_{(|x| \geq 1)} \left(\tau_{\bar{G}(X_{s-}, x)} - Id \right) \tau_{X_{s-}} \phi N(dsdx)
\end{aligned}$$

Hence $Y_t(\phi) := \tau_{X_t} \phi$ solves the equation (4.8). \square

Acknowledgement: The author would like to thank Professor B. Rajeev, Indian Statistical Institute, Bangalore for valuable suggestions during the work and pointing out the way to Theorem 4.5.

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